

## ON A THEOREM OF SCOTT AND SWARUP

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ABSTRACT. Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  be an exact sequence of hyperbolic groups induced by a fully irreducible automorphism  $\phi$  of the free group  $H$ . Let  $H_1(\subset H)$  be a finitely generated distorted subgroup of  $G$ . Then  $H_1$  is of finite index in  $H$ . This is an analog of a Theorem of Scott and Swarup for surfaces in hyperbolic 3-manifolds.

## 1. INTRODUCTION

In [16], Scott and Swarup prove the following theorem:

**Theorem** [16] *Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  be an exact sequence of hyperbolic groups induced by a pseudo Anosov diffeomorphism of a closed surface with fundamental group  $H$ . Let  $H_1$  be a finitely generated subgroup of infinite index in  $H$ . Then  $H_1$  is quasiconvex in  $G$ .*

In this paper we derive an analogous result for free groups (see Section 2 below or [3] [2] [7] for definitions).

We note at the outset that *hyperbolic* stands for two notions. When qualifying manifolds, they indicate spaces of constant curvature equal to -1. When qualifying groups or metric spaces, we use *hyperbolic* in the sense of Gromov [8]. It will be clear from the context which of these meanings is relevant.

**Theorem 3.5** *Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  be an exact sequence of hyperbolic groups induced by an aperiodic fully irreducible automorphism of the free group  $H$ . Let  $H_1$  be a finitely generated subgroup of infinite index in  $H$ . Then  $H_1$  is quasiconvex in  $G$ .*

In fact the methods of this paper can be used to give a new proof of the Theorem of Scott and Swarup mentioned above. We sketch this proof for closed surfaces first. Let  $M$  be a closed hyperbolic 3-manifold fibering over the circle with fiber  $F$ . Let  $\tilde{F}$  and  $\tilde{M}$  denote

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the universal covers of  $F$  and  $M$  respectively. Then  $\tilde{F}$  and  $\tilde{M}$  are quasi-isometric to  $\mathbb{H}^2$  and  $\mathbb{H}^3$  respectively. Now let  $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}_\infty^1$  and  $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}_\infty^2$  denote the standard compactifications. In [5] Cannon and Thurston show that the usual inclusion  $i$  of  $\tilde{F}$  into  $\tilde{M}$  extends to a continuous map  $\hat{i}$  from  $\mathbb{D}^2$  to  $\mathbb{D}^3$ . Cannon and Thurston further show that  $\hat{i}$  identifies precisely those pairs of points which are boundary points of an ending lamination. Since a leaf of the stable (or unstable) lamination is dense in the whole lamination, it cannot be carried by a (perhaps immersed) proper sub-surface (one can see this, for instance, by using the fact that surface groups are LERF [15]). The subgroup corresponding to the fundamental group of such a subsurface must therefore be quasiconvex in  $G$ .

This idea goes through for free groups. We give a brief sketch for aperiodic automorphisms. In this case, Bestvina, Feighn and Handel [3] have shown that any leaf of the stable (or unstable) lamination ‘fills’  $H$ , i.e. it cannot be carried by a finitely generated subgroup  $H_1$  of infinite index in  $H$ . We combine this with the description of boundary identifications given in [11] to show that no pair of points on the boundary of  $H_1$  are identified. Thus  $H_1$  must be quasiconvex in  $G$ .

## 2. ENDING LAMINATIONS

Let  $G$  be a hyperbolic group in the sense of Gromov [8]. Let  $H$  be a hyperbolic subgroup of  $G$ . Choose a finite generating set of  $G$  containing a finite generating set of  $H$ . Let  $\Gamma_G$  and  $\Gamma_H$  be the Cayley graphs of  $G, H$  with respect to these generating sets. Let  $i : \Gamma_H \rightarrow \Gamma_G$  denote the inclusion map.

**Definition :** [7] [6] *If  $i : \Gamma_H \rightarrow \Gamma_G$  be an embedding of the Cayley graph of  $H$  into that of  $G$ , then the distortion function is given by*

$$disto(R) = Diam_{\Gamma_H}(\Gamma_H \cap B(R)),$$

*where  $B(R)$  is the ball of radius  $R$  around  $1 \in \Gamma_G$ .*

If  $H$  is quasiconvex in  $G$  the distortion function is linear and we shall refer to  $H$  as an undistorted subgroup. Else,  $H$  will be termed distorted.

For distorted subgroups, the distortion information is encoded in a certain set of ending laminations defined below.

**Definition :** *If  $\lambda$  is a geodesic segment in  $\Gamma_H$  then  $\lambda^r$ , a **geodesic realization** of  $\lambda$ , is a geodesic in  $\Gamma_G$  joining the end-points of  $i(\lambda)$ .*

Now consider sequences of geodesic segments  $\lambda_i \subset \Gamma_H$  such that  $1 \in \lambda_i$  and  $\lambda_i^r \cap B(i) = \emptyset$ , where  $B(i)$  is the ball of radius  $i$  around

$1 \in \Gamma_G$ . Take all bi-infinite subsequential limits (in the Hausdorff topology) of all such sequences  $\{\lambda_i\}$  and denote this set by  $\Sigma$ .

Let  $t_h$  denote left translation by  $h \in H$ . Let  $\widehat{\Gamma}_H$  and  $\widehat{\Gamma}_G$  denote the Gromov compactifications of  $\Gamma_H$  and  $\Gamma_G$  respectively. Further let  $\partial\Gamma_H$  and  $\partial\Gamma_G$  denote the boundaries of  $\Gamma_H$  and  $\Gamma_G$  respectively [8].

**Definition :** *The set of ending laminations  $\Lambda = \Lambda(H, G)$  is given by*

$$\Lambda = \{(p, q) \in \partial\Gamma_H \times \partial\Gamma_H \mid p \neq q \text{ and } p, q \text{ are the end-points of } t_h(\lambda) \text{ for some } \lambda \in \Sigma\}$$

**Lemma 2.1.**  *$H$  is quasiconvex in  $G$  if and only if  $\Lambda = \emptyset$*

**Proof :** Suppose  $H$  is quasiconvex in  $G$ . Then any geodesic realization  $\lambda^r$  of a geodesic segment  $\lambda \subset \Gamma_H$  lies in a bounded neighborhood of  $\Gamma_H$  and hence of  $\lambda$  as  $H$  is hyperbolic. Hence  $\Lambda = \emptyset$ .

Conversely, if  $H$  is not quasiconvex in  $G$ , there exist  $\lambda_i \subset \Gamma_H$  and  $p_i \in \lambda_i$  such that  $\lambda_i^r \cap B_{p_i}(i) = \emptyset$ , where  $B_{p_i}(i)$  denotes the ball of radius  $i$  around  $p_i$  in  $\Gamma_G$ . Translating by  $p_i^{-1}$  and taking subsequential limits, we get  $\Sigma \neq \emptyset$  and hence  $\Lambda \neq \emptyset$ .  $\square$

**Definition :** *A Cannon-Thurston map for the pair  $(H, G)$  is a map  $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$  which is a continuous extension of  $i : \Gamma_H \rightarrow \Gamma_G$ .*

Note that if such a continuous extension exists, it is unique. We get a simplified collection of ending laminations when a Cannon-Thurston map exists.

**Definition :**  $\Lambda_{CT} = \{(p, q) \in \partial\Gamma_H \times \partial\Gamma_H \mid p \neq q \text{ and } \hat{i}(p) = \hat{i}(q)\}$ .

**Lemma 2.2.** *If a Cannon-Thurston map exists,  $\Lambda = \Lambda_{CT}$ .*

**Proof:** Let  $(p, q) \in \Lambda$ . After translating by an element of  $H$  if necessary assume that a bi-infinite geodesic  $\lambda$  passing through 1 has  $p, q$  as its end-points. By definition of  $\Lambda$  there exist geodesic segments  $\lambda_i \subset \Gamma_H$  converging to  $\lambda$  in the Hausdorff topology such that  $\lambda_i^r \cap B(i) = \emptyset$ . Since a Cannon-Thurston map exists, there exists  $z \in \partial\Gamma_G$  such that  $\lambda_i^r \rightarrow z$  in the Hausdorff topology on  $\widehat{\Gamma}_G$  and  $\hat{i}(p) = z = \hat{i}(q)$ . Hence  $\Lambda \subset \Lambda_{CT}$ .

Conversely, let  $(p, q) \in \Lambda_{CT}$ . After translating by an element of  $H$  if necessary assume that a bi-infinite geodesic  $\lambda$  passing through 1 has  $p, q$  as its end-points. Choose  $p_i, q_i \in \Gamma_H$  such that  $p_i \rightarrow p$  and  $q_i \rightarrow q$ . Let  $\lambda_i$  denote the subsegment of  $\lambda$  joining  $p_i, q_i$ . Then  $\lambda_i^r$  converges to  $\hat{i}(p) = \hat{i}(q)$  in the Hausdorff topology on  $\widehat{\Gamma}_G$ . Passing to a subsequence if necessary we can assume that  $\lambda_i^r \cap B(i) = \emptyset$ . Hence  $\Lambda_{CT} \subset \Lambda$ .  $\square$

**Remark:** Suppose  $H_1$  is a hyperbolic subgroup of  $H$ . Let  $\hat{j}$  and  $\hat{i}$  denote Cannon-Thurston maps for the pairs  $(H_1, H)$  and  $(H, G)$  respectively. Then the composition  $\hat{i} \cdot \hat{j}$  is a Cannon-Thurston map for the pair  $(H_1, G)$ . Further from Lemma 2.2 it follows that

$$\Lambda(H_1, G) = \Lambda(H_1, H) \cup (\hat{j})^{-1}(\Lambda(H, G)).$$

### 3. EXTENSIONS BY FREE GROUPS

Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  denote an exact sequence of hyperbolic groups arising out of a hyperbolic automorphism  $\phi$  of the hyperbolic group  $H$ . The notion of a hyperbolic automorphism was defined in [1] (see below) and shown to be equivalent to requiring that  $G$  be hyperbolic.

**Definition:** Let  $\phi$  be an automorphism of a hyperbolic group  $H$  (equipped with the word metric  $|\cdot|$ ). Let  $\lambda > 1$ . Let  $S(\phi, \lambda) = \{h \in H : |\phi(h)| > \lambda|h|\}$ . If  $h \in S(\phi, \lambda)$ , we say  $\phi$  stretches  $h$  by  $\lambda$ .  $\phi$  will be called **hyperbolic** if for all  $\lambda > 1$  there exists  $n > 0$  such that for all  $h \in H$ , at least one of  $\phi^n$  or  $\phi^{-n}$  stretches  $h$  by  $\lambda$ .

$\phi$  and  $\phi^{-1}$  induce bijections (also denoted by  $\phi$  and  $\phi^{-1}$ ) of the vertices of  $\Gamma_H$ .

A **free homotopy representative** of a word  $w \in H$  is a geodesic  $[a, aw_0]$  in  $\Gamma_H$  where  $w_0$  is a shortest word in the conjugacy class of  $w$  in  $H$ .

Given  $h \in H$  let  $\Sigma(h, n, +)$  (resp.  $\Sigma(h, n, -)$ ) be the ( $H$ -invariant) collection of all free homotopy representatives of  $\phi^n(h)$  (resp.  $\phi^{-n}(h)$ ) in  $\Gamma_H$ . The intersection with  $\partial\Gamma_H \times \partial\Gamma_H$  of the union of all bi-infinite subsequential limits (in the Hausdorff topology on  $\widehat{\Gamma_H}$ ) of elements of  $\Sigma(h, n, +)$  (resp.  $\Sigma(h, n, -)$ ) as  $n \rightarrow \infty$  will be denoted by  $\Lambda_+(h)$  (resp.  $\Lambda_-(h)$ ).

**Definition:** The **stable** and **unstable ending laminations** are respectively given by

$$\begin{aligned} \Lambda_+ &= \bigcup_{h \in H} \Lambda_+(h) \\ \Lambda_- &= \bigcup_{h \in H} \Lambda_-(h) \end{aligned}$$

**Theorem 3.1.** [13] Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  denote an exact sequence of hyperbolic groups arising out of a hyperbolic automorphism  $\phi$  of the hyperbolic group  $H$ . Then there exists a Cannon Thurston map for the pair  $(H, G)$ .

**Theorem 3.2.** [11] *Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  denote an exact sequence of hyperbolic groups arising out of a hyperbolic automorphism  $\phi$  of the hyperbolic group  $H$ . Then  $\Lambda_{CT} = \Lambda_+ \cup \Lambda_-$ .*

Further, it is shown in [11] that only finitely many  $h$ 's need be considered in the definition of  $\Lambda_+$  or  $\Lambda_-$ .

We turn now to the main focus of this paper, the case where  $H$  is free and  $\phi$  is an irreducible hyperbolic automorphism [1]. Such automorphisms have been studied in great detail by Bestvina, Feighn and Handel [1], [3], [2].

**Definition:** A non negative irreducible matrix is *aperiodic* if it has an iterate that is strictly positive.

**Definition:** Let us assume for a start that transition matrices of  $\phi$  and  $\phi^{-1}$  with respect to train-track representatives [see [4] for definitions] are aperiodic. We shall refer to such automorphisms as *aperiodic*. Note that in this definition, we require transition matrices of both  $\phi$  and  $\phi^{-1}$  to be aperiodic.

In this case, the definitions of ending laminations here and in [3] do not coincide. However the difference between these is now understood [9, 10]. We recall definitions from [3].

Let  $f : X \rightarrow X$  be a train-track representative of an outer automorphism with aperiodic transition matrix. Endow  $X$  with the structure of a marked  $\mathbb{R}$ -graph so that  $f$  expands lengths of edges by a uniform factor  $\lambda > 1$ . Let  $x \in X$  be an  $f$ -periodic point in the interior of some edge. Let  $\epsilon > 0$  be small, and let  $U$  be the  $\epsilon$ -neighborhood of  $x$ . Then for some  $N > 0$ ,  $U \subset f^N(U)$ . Choose an isometry  $l : (-\epsilon, \epsilon) \rightarrow U$  and extend it to the unique locally isometric immersion  $l : \mathbb{R} \rightarrow X$  such that  $l(\lambda^N t) = f^N(l(t))$ . We say  $l$  is obtained by iterating a neighborhood of  $x$ .  $l$  will also be termed a *leaf* of the ending lamination.

**Definitions :** Two isometric immersions  $[a, b] \rightarrow X$  and  $[c, d] \rightarrow X$  are said to be equivalent if there is an isometry of  $[a, b]$  onto  $[c, d]$  making the triangle commute.

A *leaf segment* of an isometric immersion  $\mathbb{R} \rightarrow X$  is the equivalence class of the restriction to a finite interval. A *half leaf* of an isometric immersion  $\mathbb{R} \rightarrow X$  is the equivalence class of the restriction to a semi-infinite interval  $(-\infty, a]$  or  $[b, \infty)$ .

Two isometric immersions  $l, l' : \mathbb{R} \rightarrow X$  are *(weakly) equivalent* if every leaf segment of  $l$  is a leaf segment of  $l'$  and vice versa.

Since  $f$  has an aperiodic transition matrix,  $l$  is surjective. Using this, Bestvina, Feighn and Handel [3] show that any two leaves of the ending lamination obtained by iterating neighborhoods of  $f$ -periodic points are

equivalent. Let  $\Lambda_{BFH}(f)$  denote the collection of leaves obtained from  $f$  in this way.

**Relation Between  $\Lambda_{BFH}(f)$  and Ending Laminations**<sup>1</sup>

$Diag(\Lambda_{BFH}(f))$  will denote the diagonal extension of  $\Lambda_{BFH}(f)$  following [9]. In the context of surfaces this simply corresponds to adding on the diagonals of an ideal polygon. We now explain what this is more precisely in the context of free groups. Define a relation  $p \sim q$ , if  $p, q$  are end-points of a leaf of  $\Lambda_{BFH}(f)$ . Then the transitive closure of the relation of this relation gives  $Diag(\Lambda_{BFH}(f))$ , i.e.  $p, q$  are end-points of a leaf of  $Diag(\Lambda_{BFH}(f))$  if there exists a finite sequence  $p = p_0, p_1, \dots, p_n = q$  such that  $p_i, p_{i+1}$  are end-points of a leaf of  $\Lambda_{BFH}(f)$  for all  $i = 0, \dots, n-1$ .

The following Proposition due to Kapovich and Lustig furnishes the relationship we need between  $\Lambda_{BFH}(f)$  and  $\Lambda_+$ .

**Theorem 3.3.** (*Proposition 6.4 of [10]*) *Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  be an exact sequence of hyperbolic groups induced by an aperiodic automorphism  $\phi$  of the free group  $H$  and let  $f$  be a train-track representative of  $\phi$ . Then  $Diag(\Lambda_{BFH}(f)) = \Lambda_+$ .*

We can now use the results of [3] and [2] in our context.

**Remark :** Since any two leaves are (weakly) equivalent in the sense of [3] above, the equivalence class can alternately be obtained by translating some (any) leaf by elements of the free group and taking Hausdorff limits. This is analogous to the case for surfaces where the stable lamination of a pseudo anosov diffeomorphism is the closure of some (any) leaf.

The next Proposition follows from [3] (see particularly Propositions 1.6 and 2.4) and Theorem 3.3. It says roughly that any leaf of the stable (or unstable) lamination of an aperiodic automorphism ‘fills’ the free group  $H$ .

The proof of Proposition 1.6 of [3] shows that any *half-leaf* ‘fills’ the free group  $H$ . Proposition 3.3 shows that any leaf of the stable (or unstable) lamination must contain a *half-leaf* of the corresponding lamination  $\Lambda_{BFH}(f)$ .

**Proposition 3.4.** *Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  be an exact sequence of hyperbolic groups induced by an aperiodic automorphism  $\phi$  of the free group  $H$  (i.e.  $\phi$  and  $\phi^{-1}$  have aperiodic transition matrices). If  $(p, q) \in$*

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<sup>1</sup>I had incorrectly equated  $\Lambda_+$  and  $\Lambda_{BFH}(f)$  at this stage of [14]. The difference lies in the diagonal leaves to be added.

$\Lambda_+$  or  $\Lambda_-$  lie in the boundary  $\partial\Gamma_{H_1} \subset \partial\Gamma_H$  for a finitely generated subgroup  $H_1$  of  $H$ , then  $H_1$  is of finite index in  $H$ .

*Proof.* It suffices to show that no leaf of  $\Lambda_+$  (or  $\Lambda_-$ ) is carried by a finitely generated subgroup  $H_1$  of infinite index  $H$ . As noted above, the proof of Proposition 1.6 of [3] shows that any *half-leaf* ‘fills’ the free group  $H$ , i.e. it cannot have a limit point in a translate of the limit set of  $H_1$ . Hence no end-point of a leaf of  $\Lambda_{BFH}(f)$  for a train-track representative  $f$  of  $\phi$  can have a limit point in a translate of the limit set of  $H_1$ . Since the set of end-points of  $\Lambda_{BFH}(f)$  coincide with the end-points of  $\Lambda_+$  by Proposition 3.3 we are done.  $\square$

We are now in a position to prove the main theorem of this paper for aperiodic  $\phi$ .

**Theorem 3.5.** *Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  be an exact sequence of hyperbolic groups induced by an aperiodic automorphism  $\phi$  of the free group  $H$ . Let  $H_1$  be a finitely generated subgroup of infinite index in  $H$ . Then  $H_1$  is quasiconvex in  $G$ .*

**Proof:** From Theorem 3.1 the pair  $(H, G)$  has a Cannon - Thurston map. Further, from Theorem 3.2

$$\Lambda_{CT}(H, G) = \Lambda(H, G) = \Lambda_+ \cup \Lambda_-.$$

Let  $j : H_1 \rightarrow H$  and  $i : H \rightarrow G$  denote inclusions. Since  $H_1$  is quasiconvex in  $H$ ,  $\Lambda(H_1, H) = \emptyset$  (Lemma 2.1). Further from Proposition 3.4 above  $j^{-1}(\Lambda(H, G)) = \emptyset$ .

Also from the remark following Lemma 2.2,  $\Lambda(H_1, G) = \Lambda(H_1, H) \cup j^{-1}(\Lambda(H, G)) = \emptyset$ .

Hence from Lemma 2.1  $H_1$  is quasiconvex in  $G$ .  $\square$

As a second step we deal with automorphisms  $\phi$  of  $H$  satisfying the following:

There exists a decomposition  $H = K_1 * K_2 * \cdots * K_n$  of  $H$  into  $\phi$ -invariant factors  $K_i$  such that the restrictions  $\phi|_{K_i} = \phi_i$  are aperiodic.

Let  $\Lambda_i$  denote the ending laminations of  $\phi_i$ .

We need to first show that the endpoints of *half-leaves* of  $\Lambda \cap \partial\Gamma_{K_i}$  are precisely the endpoints of *half-leaves*  $\Lambda_i$ . Let us consider a reduced word  $w = a_1 \cdots a_k$  such that each  $a_i$  is a maximal subword lying in a  $\phi$ -invariant factor  $K_i$ . We now consider subsequential limits of the geodesic representatives of  $\phi^n(w)$  giving us the leaves of  $\Lambda_+$ . Since each  $a_i$  is a maximal subword lying in a  $\phi$ -invariant factor, there is no cancellation between the geodesic representatives of  $\phi^n(a_i)$ . Also note that since the lengths of each  $\phi^n(a_i)$  tends to infinity as  $n$  tends to infinity, it suffices to consider  $k = 1, 2$ . If  $k = 1$ , then the subsequential

limits of the iterates  $\phi^n(a_1)$  are precisely the leaves of  $\Lambda_{i+}$  and we are done in this case. Let  $k = 2$  and  $w = a_1a_2$ . Then the subsequential limits of the iterates  $\phi^n(w) = \phi^n(a_1)\phi^n(a_2)$  contain half leaves of limits of the iterates  $\phi^n(a_1)$  or  $\phi^n(a_2)$  or both. In any case the endpoints of *half-leaves* of  $\Lambda \cap \partial\Gamma_{K_i}$  are precisely the endpoints of *half-leaves*  $\Lambda_i$ .

Suppose  $H_1$  is a finitely generated subgroup of  $H$  that is distorted in  $G$ . Since  $H_1$  is quasi-convex in  $H$ , there exists a pair  $(p, q) \in \Lambda = \Lambda_+ \cup \Lambda_-$  lying on the boundary  $\partial\Gamma_{H_1} \subset \partial\Gamma_H$ . Let  $l$  be a leaf of  $\Lambda$  joining  $p, q$ . By Theorem 3.2  $l$  lies in the Hausdorff limit of sequences of segments obtained by iterating  $\phi$  or  $\phi^{-1}$  on some  $h \in H$ . By the pigeon-hole principle there exists arbitrarily long segments of  $l$  contained in a (fixed) conjugate of  $K_i$  for some  $i$ . For ease of exposition let us assume that this is the trivial conjugate of  $K_i$ , i.e.  $K_i$  itself. Translating by appropriate elements of  $H_1$  and taking a Hausdorff limit we obtain an endpoint of a leaf (or half-leaf) of the ending lamination  $\Lambda$  lying in the intersection  $\partial\Gamma_{H_1} \cap \partial\Gamma_{K_j}$ . In particular, there exists a point  $s \in \partial\Gamma_{H_1}$  which is also an end-point of a half-leaf of  $\Lambda_j$  (by the argument in the previous paragraph).

Since intersection of quasiconvex subgroups is quasiconvex [17] it follows that  $H_1 \cap K_j$  is quasi-convex in  $H$ . In particular  $H_1 \cap K_j$  is finitely generated. Also as observed above,  $H_1$  has a boundary point in common with a leaf of the ending lamination  $\Lambda_j$ . Hence from Theorem 3.5  $H_1 \cap K_j$  is a finite index subgroup of  $K_j$ .

We have shown :

**Theorem 3.6.** *Let  $1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  be an exact sequence of hyperbolic groups induced by an automorphism  $\phi$  of the free group  $H$  satisfying the following :*

*There exists a decomposition  $H = K_1 * K_2 * \cdots * K_n$  of  $H$  into  $\phi$ -invariant factors  $K_i$  such that the restrictions  $\phi|_{K_i} = \phi_i$  are aperiodic.*

*Let  $H_1$  be a finitely generated subgroup of infinite index in  $H$  such that  $H_1$  is distorted in  $G$ . Then there exist  $h \in H$  and  $K_j$  such that  $H_1$  contains a finite index subgroup of  $h^{-1}K_jh$ .*

#### 4. ERRATUM

In Theorem 3.6 in [14] I had misquoted Corollary 4.7 of [2]. Hence Theorem 3.7 in [14] stands non-proven.

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